RADCOM TECHNICAL FEATURE

An Impedance Diagram for Transmission Lines

THE CURRENT AND voltage on an unmatched transmission line will vary along the line. The ratio of the voltage between the lines to the current flowing in them is termed the *line impedance*, Z. This ratio varies from point to point, as also does the phase difference between voltage and current. The value of Z at every point is ultimately determined by the load which terminates the line, which in most cases will be the antenna. Reflection at this termination always takes place in such a way as to ensure that Z at this point is identical to the external load. The line impedance at the other end is the impedance which is presented to the transmitter.

The line impedance Z must not be confused with the *characteristic impedance* of the line, Z_0.

The problem dealt with here is how to predict the variation of impedance along an unmatched line, and thus describe how the line acts as a transformer between the transmitter and the antenna, and why this transforming action is critically dependent upon the length of the line.

The usual method of tackling this sort of problem is to employ a Smith chart and those who are familiar with its use will find that the operation rules for the diagram are very similar. On the other hand, there is no need to know anything about a Smith chart. The diagram is an alternative approach. No special chart or equipment is needed: simply a ruler, a protractor, and a pocket calculator which gives SIN, COS and TAN.

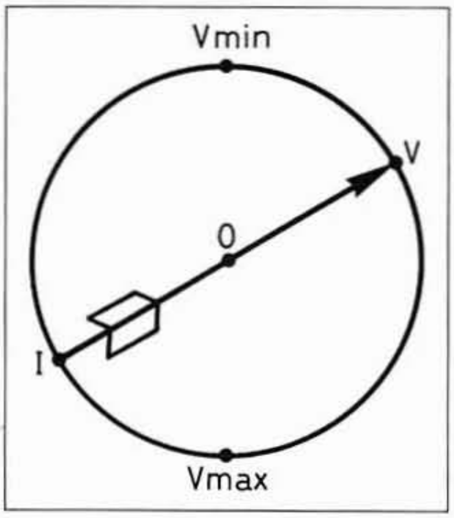

Fig 1: The basis of the diagram.

In fact the ability to predict numerical results is perhaps less important than the insight into transmission line behaviour which can be gained by sketching a few rough diagrams, without the necessity of making any measurements or calculations.

Geoffrey Billington, G3EAE, shows how a simple diagram can tell you nearly everything you need to know about the distribution of voltage, current and impedance on a transmission line.

THE DIAGRAM

PICTURE A CLOCK FACE. The two hands have been replaced by a single pointer, the 'indicator', pivoted at its centre, O. The arrowhead end of this pointer is labelled V and the tail end I (see **Fig 1**).

The dial has two points marked on it: 12 o'clock is marked V_{min}, and 6 o'clock is marked V_{max}. The impedance at any point on the line may be found by setting the indicator to the appropriate position and then incorporating it into a diagram.

If, for instance, you wish to investigate conditions at a voltage maximum, the indicator is turned until the arrowhead points to V_{max} (6 o'clock). It is turned to 12 o'clock for a voltage minimum, and to 3 o'clock or 9 o'clock to find the impedance half way between a minimum and a maximum, and so on.

If you look at Fig 1 you can see that the indicator is set for a point one-third of the distance between a voltage minimum and a voltage maximum (ie the indicator points to 2 o'clock).

In order to find the impedance at this point we need to know (i) the characteristic impedance of the line (Z_0) and (ii) the SWR.

Suppose that the SWR on the line had been measured and found to be 3.0. Knowing the SWR, we use it to calculate the reflection coefficient k which is the number we actually use in drawing the diagram.

$$k = (SWR - 1)/(SWR + 1) \quad \text{Note 1}$$
$$k = (3 - 1)/(3 + 1)$$
$$k = 0.5$$

The diagram is now constructed as follows (see **Fig 2**):

Draw a vertical line - the 'main axis' - down the centre of a sheet of paper. Mark the point 'O', the centre of the 'clock face' somewhere on the lower part of the line. Next mark in another point 'P' on the line, at some convenient known distance (say 100mm) above O.

The main axis will pass through 12 o'clock and 6 o'clock on the dial when this is added later.

Next work out the length of the indicator or, more conveniently, the half length 'OV' (or 'OI'). This distance must be made equal to k x OP, eg to 100k millimetres if OP has been drawn 100mm long as suggested. The actual scale used is immaterial.

In this example the half length will be:

$$0.5 \times 100 = 50\text{mm}.$$

The clock face, radius 50mm, may now be

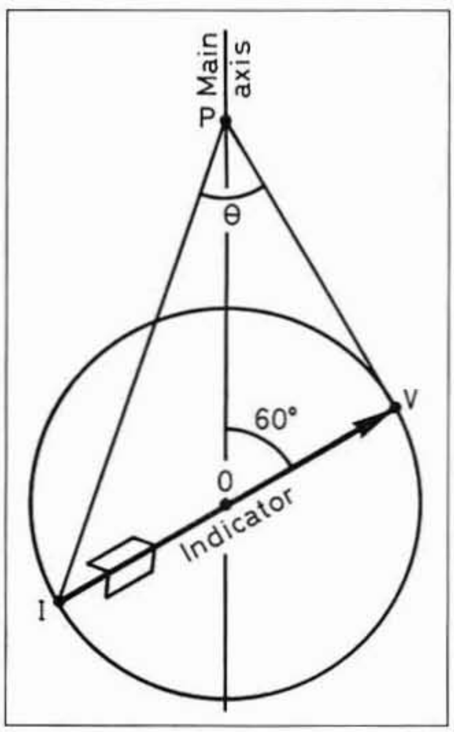

Fig 2: The diagram in use.

roughly sketched in (Note 2), and the indicator drawn accurately to size and making the correct angle with the main axis. As the indicator should point at 2 o'clock, the angle made with the main axis is 180/3 = 60°. The diagram is completed by joining the tip and the tail of the indicator to point P (see **Fig 2**).

FINDING Z

TO FIND Z, MEASURE PV and PI and work out (PV/PI).

Then $Z/Z_0 = (PV/PI)$
Or $Z = Z_0(PV/PI)$

In the given example (**Fig 2**), (PV/PI) = 0.655, so the impedance is 0.655 multiplied by the characteristic impedance of the line. For 50Ω coaxial cable, for instance, the impedance at this point will be 32.75Ω, say 33Ω.

However, knowing Z is not a lot of use. What is needed for practical purposes is the value of the resistive component (R) and reactive component (X). To find these quantities, the angle θ between PV and PI must be measured (Note 3).

In the example shown, θ = 49°. R and X are then found as follows: First find Z as previously explained.

$$Z = Z_0(PV/PI) = 33\Omega$$
$$R = Z\cos\theta = 33\cos 49 = 22\Omega$$
$$X = Z\sin\theta = 33\sin 49 = 25\Omega$$

Z is equivalent to a resistance R and a reactance X connected in series (whether X is capacitive or inductive is explained later.)

In order to understand the above equations better, picture what happens when the indicator is rotated. The lengths PV and PI change, showing that Z changes, and the angle between them (θ) opens out and then closes up again.

Put non-mathematically, the greater the angle θ, the greater is the reactive component (X) compared to the resistive component (R). When θ = 0 the reactive component is zero and Z is a pure resistance.

When the indicator is set to V_{max} or to V_{min}, PV and PI lie together, so θ = 0 at these two points and Z is a pure resistance. These are the only points on an unmatched line where this occurs. At all other points Z has a reactive as well as a resistive component. Picturing how PV and PI change as the indicator rotates also shows that the line impedance has its maximum and minimum values at a voltage maximum and voltage minimum respectively.

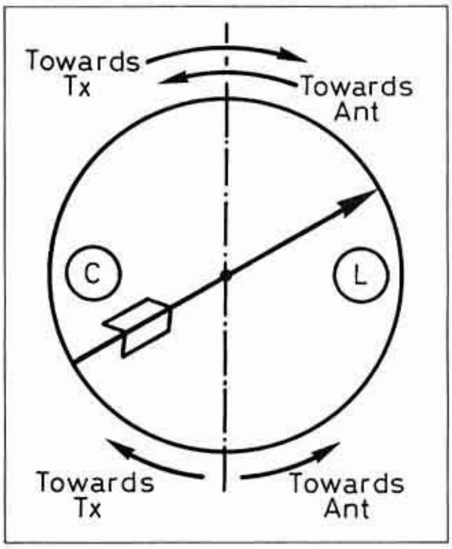

Fig 3: Another rule must be obeyed.

Note that because θ = 0 at a voltage maximum or minimum, the equations give

$$R = Z\cos 0 = Z \quad (\cos 0 = 1)$$
$$X = Z\sin 0 = 0 \quad (\sin 0 = 0)$$

This agrees with the statement above that at a voltage maximum or minimum, Z is a pure resistance with zero reactance.

The only remaining question is whether X is inductive or capacitive. This is easily settled.

Put the letter 'L' on the right hand side of the main axis, and 'C' on the left. If the arrowhead of the indicator lies to the right of the main axis, the reactive component is inductive, and if the arrow lies on the left it is capacitive.

In order that this rule works another rule must be obeyed: movement along the line *away* from the transmitter end of the feeder is represented by an *anticlockwise* rotation of the indicator. Movement along the line *towards* the transmitter is represented by a *clockwise* rotation. The rules are summarised in **Fig 3**. (These same rules are employed when using a Smith chart).

Before explaining how to use the diagram to solve numerical problems, it is worthwhile looking at a few of the many ways in which it can shed light on transmission line behaviour.

THE MATCHED LINE (K = 0)

IF A LINE IS MATCHED by termination in a pure resistance equal to its characteristic impedance Z_0, there will be no reflection from the termination, ie the SWR = 1, and k = 0.

To demonstrate how this affects the diagram it is easiest to consider first what happens when k is very small. It is then easy to see what will happen if k is made smaller and smaller, eventually becoming zero.

Fig 4 shows an example where the SWR = 1.2 and k = 0.09. It is clear that whatever the position of the indicator, PV/PI is never very different from unity, and the angle between PV and PI is always small. This means that the impedance for any length of line is close to Z_0 and has a negligible reactive component.

In the limiting case of k = 0, the length of the indicator dwindles to zero, so PV and PI are always the same length, and always lie one on top of the other: Z is always equal to Z_0, and is always a pure resistance. The line is matched.

IMPEDANCES AT VOLTAGE MAXIMA AND MINIMA

NEXT CONSIDER WHAT THE diagram has to say about the impedances (resistances) at the maxima and minima. In **Fig 5** OP is taken as one unit long, and the indicator half length is therefore k units.

At a voltage maximum

$$PV = (1 + k) \text{ and } PI = (1 - k)$$

$$Z/Z_0 = (1 + k)/(1 - k)$$

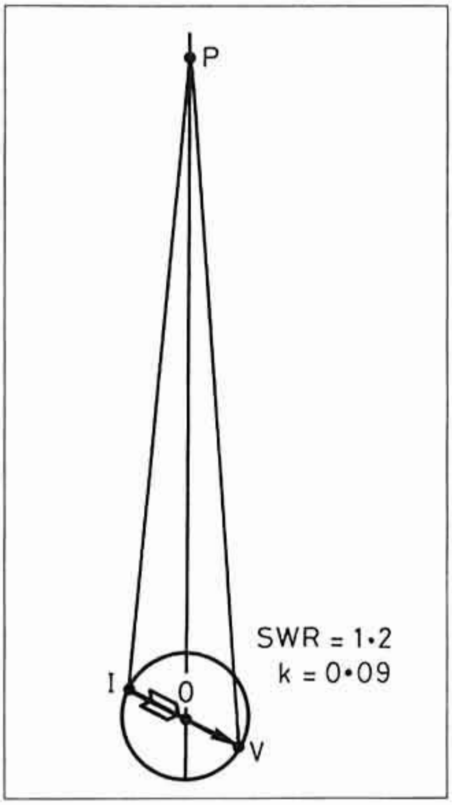

Fig 4: An example.

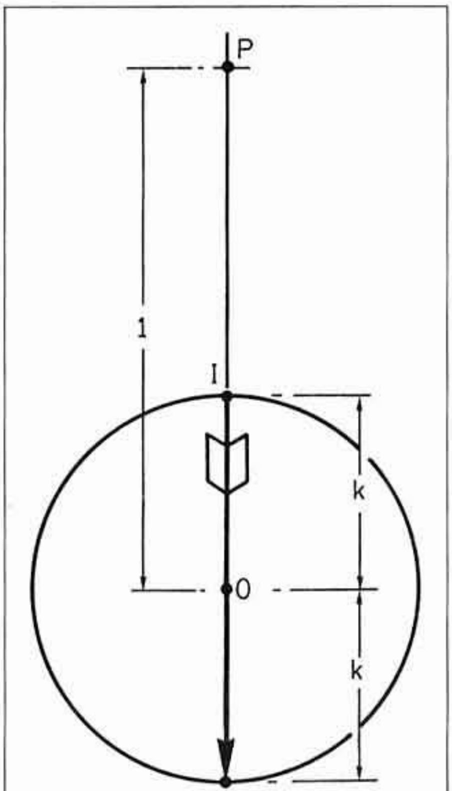

Fig 5: Impedance as a sure resistance.

But (1 + k)/(1 - k) = SWR (As can be shown by rearranging the previous equation for k). So at a voltage maximum the impedance is a pure resistance R given by

$$R/Z_0 = SWR$$

Conversely, when a line is terminated in a pure resistance R which is greater than Z_0

there will be a voltage maximum at the termination and the SWR will be equal to R/Z_0.

Applying similar arguments to a voltage minimum you can show that the impedance (again a pure resistance) is equal to Z_0/SWR, or conversely, if the line is terminated with a resistance R which is less than Z_0, there will be a voltage minimum at the termination and the SWR will be equal to Z_0/R.

PERFECTLY REFLECTING TERMINATIONS (K = 1)

K IS EQUAL TO UNITY (and the SWR is infinite) when a line is terminated by a perfect reflector which absorbs no energy, for instance an open-circuit, a short-circuit or a pure reactance. **Figs 6(a)** and **6(b)** illustrate cases where k = 1. OP is equal to the half length of the indicator, so 'P' is situated on the rim of the 'dial', and because of this it turns out that *the angle θ is a right angle for all settings of the indicator* (Note 4).

Since sin 90 = 1 and cos 90 = 0, Z is always a pure reactance with no resistive component. A length of open-circuit or short-circuited transmission line behaves as either a capacitor or an inductor, depending upon its length.

Now think about what happens at a voltage minimum. As the indicator approaches 12 o'clock, the length PV dwindles to zero. Thus (PV/PI) becomes zero at this point, and so does Z. Fairly obviously, this must apply at the short-circuit termination.

In a similar way PI becomes zero at a voltage maximum which means that (PV/PI) becomes infinite and so does Z. An open-circuit termination must be a voltage maximum.

The impedance (reactance) of any length of open-circuit or short-circuit length of line may be found by setting the indicator to 12 o'clock for a short-circuit, or 6 o'clock for an open-circuit, and rotating the indicator clockwise through the angle which represents the length of line.

The reader may feel unhappy about the prediction that for lines terminated in a perfect reflector, the impedance will be infinite at a voltage maximum and zero at a voltage minimum. Clearly this can never quite happen in practice. There must always be some slight loss of energy on the line or at the termination and this will prevent the ideal state from being realised.

A more realistic diagram is obtained if k is assumed to be very slightly less than unity. In this case the point P will lie just outside the circle as illustrated in **Fig 7**.

At 12 o'clock and at 6 o'clock PV and PI swing into line and the impedance converts into a pure resistance; a very high resistance at the voltage maximum and a very low resistance at the voltage minimum. At points not in the vicinity of a maximum or a minimum the diagram is almost indistinguishable from the case when k = 1. The angle between PV and PI is nearly a right angle, so Z is very nearly a pure reactance.

This clearly explains how open-circuit and short-circuit lines can exhibit resonance, in the same sort of way as a tuned circuit containing a coil and capacitor.

If it is arranged that an open-circuit or short-circuit line presents a voltage maximum to a generator, the behaviour of the line is very similar to that of a parallel-tuned circuit, with the impedance peaking up to a high resistive value at resonance. The impedance will be capacitive on one side of resonance, and inductive on the other. Similarly, if the line presents a voltage minimum to the generator, it behaves as a series-tuned circuit.

Of course the main difference between resonant lines and tuned circuits is that a simple LC circuit may (ideally) only have one resonant frequency. This can never be the case with transmission lines, where resonance can be obtained with any frequency which fulfils the required condition of a voltage maximum or minimum at the driven end.

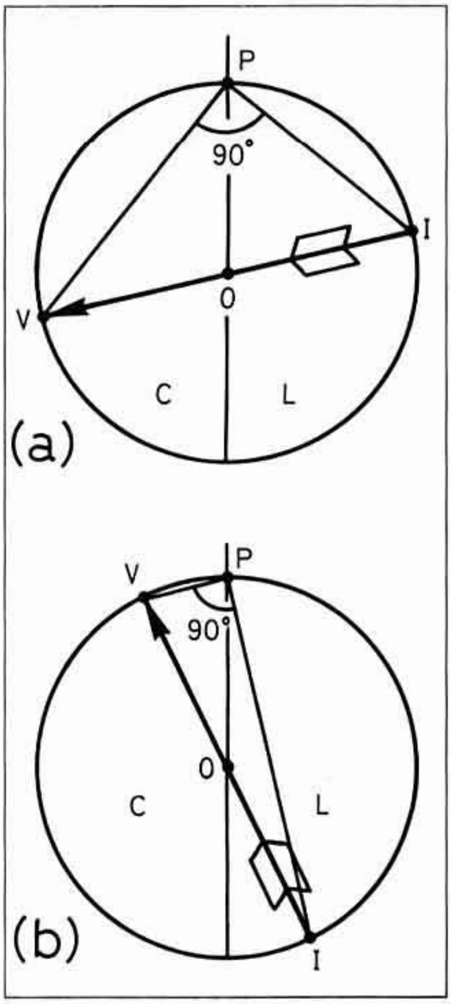

Fig 6: Perfectly reflecting terminations.

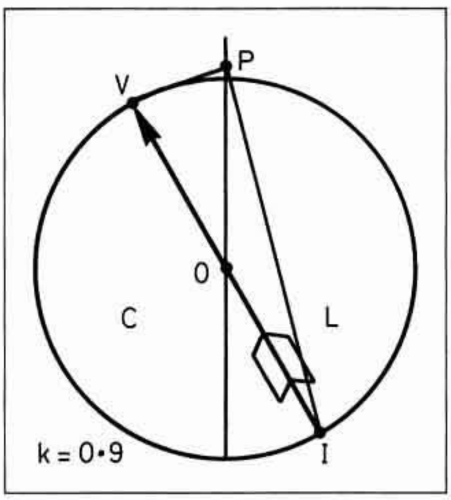

Fig 7: k is slightly less than unity.

FINDING THE ROTATION

ALL THE PREVIOUS results are of a general nature, and have been obtained without drawing any diagrams to scale or making measurements. Before applying the method to numerical problems, more should be said about determining the angle through which the indicator must be moved. So far, all discussion has been in terms of wavelength, but it is more convenient if the rotation angle can be found from the frequency of the transmitter.

It can be shown that the rotation (degrees) representing a length of L metres of line when the frequency is f megahertz is given by:

Angular rotation (degrees)
= 2.4 Lf/(velocity factor)

This may be derived as follows. A movement of one wavelength is represented by turning the indicator through *two* complete rotations: 720°. (The reason for the unexpected factor of two is mentioned in the final section: 'Meaning of the Diagram and its Limitations').

In general, a length L is represented by a rotation of 720 (L/λ) degrees. λ is the 'wavelength on the line'. This is usually less than the free space wavelength and is given by

λ = (300 x velocity factor)/f

where λ is in metres, and f in megahertz. Combining these two formulas gives the formula stated above:

Angular rotation (degrees)
= 2.4 Lf/(velocity factor)

The velocity factor depends upon the type of line used. For open-wire feeders it may be taken as unity, which means that you can forget it. For many common solid dielectric types of coaxial cable it is 0.66, though the figure for cable with air spaces in the dielectric is likely to be higher, and must be ascertained.

NOTES

Note 1: There are two versions of this formula:

(i) k = (SWR - 1)/SWR + 1)

Rearranging this gives:

(ii) SWR = (1 + k)/(1 - k).

Both versions are used in this article.

Note 2: There is no real necessity to draw the clock face, but it's not a bad idea to sketch it in.

Note 3: θ is in fact the phase difference between the line voltage and current.

Note 4: This is always true for the angle between two lines drawn from a point on the circumference to the ends of a diameter (the indicator in this case).

NEXT MONTH . .

IN PART 2, G3EAE goes through a worked example.

An Impedance Diagram for Transmission Lines

Geoffrey Billington, G3EAE, concludes with some practical solutions to impedance matching problems

TO USE THE DIAGRAM to solve a specific problem there must be sufficient information available to draw the indicator of the correct length and in the correct position for one point on the line. To find the impedance at another point the indicator must be rotated to its new position and a diagram then constructed. The following problem illustrates some of the points which have been introduced.

An antenna has a feed-point impedance which is a pure resistance of 50Ω at the operating frequency of 7.1MHz. It is fed with a 15m length of 75Ω coaxial cable, velocity factor 0.66. Find the impedance presented to the transmitter.

As the termination is a pure resistance of 50Ω and the characteristic impedance of the line is 75Ω, we know:

(i) The antenna end of the line is a voltage minimum, and

(ii) The SWR is 75/50 = 1.5

This gives

$$k = (SWR - 1)/(SWR + 1) = 0.5/2.5 = 0.2$$

As the antenna end of the feeder is a voltage minimum the initial position of the indicator is at 12 o'clock. The angle through which the indicator must be rotated is

$$2.4\ Lf/0.66 = (2.4 \times 15 \times 7.1)/0.66 = 387°$$

where L is the length of line (metres) between the antenna and the transmitter.

The direction of rotation is *clockwise* as we want to find the transformation produced by moving *towards the transmitter*. The diagram is shown in **Fig 8**.

OP is drawn any convenient length (say 100mm). Each half of the indicator is then 20mm long. A rotation of 387° is one full rotation of 360° plus a further 27°. All that need be done is to rotate the indicator through 27° (clockwise, from 12 o'clock). From the diagram

PV = 82.5mm
PI = 118.5mm

so PV/PI = 0.696 and θ = 11°.

Z = 0.696 x 75 = 52.2Ω

Resistive component R = 52.2 cos 11= 51.2Ω

Reactive component X = 52.2 sin 11 = 10.0Ω

The indicator arrowhead is on the *right* of OP so the reactance is inductive.

Note that the diagram may equally well be used to find the equivalent *parallel* combination of resistance and reactance. This is explained later in section (8) of the formula summary.

If the figures for the feeder and antenna had been reversed, ie if a 50Ω line had been used with a 75Ω resistive load, the load would have been greater than the characteristic impedance of the line so there would be a voltage maximum at the load, though the SWR and k would have the same values as before. The indicator would be the same length, but its initial position would be at 6 o'clock instead of at 12 o'clock.

When the method is going to be applied to open wire feeders, it may be possible to locate a maximum or minimum point at a measurable distance from the end of the feeder, using some sort of probe. The initial position of the indicator may then be set for this point if desired. If both a minimum and a maximum point are available, and if the probe has a linear response, the ratio of the maximum to the minimum reading gives the SWR, and you then have the information to find both the impedance presented to the transmitter and the antenna impedance.

DRAWING THE DIAGRAM 'IN REVERSE'

SOMETIMES THE RESISTIVE and reactive components of the line impedance at the transmitter end of the feeder may be determined using a bridge. It is still possible to draw a diagram in reverse which will fix the length and setting of the indicator at this point. Once this has been done, a rotation from this posi-

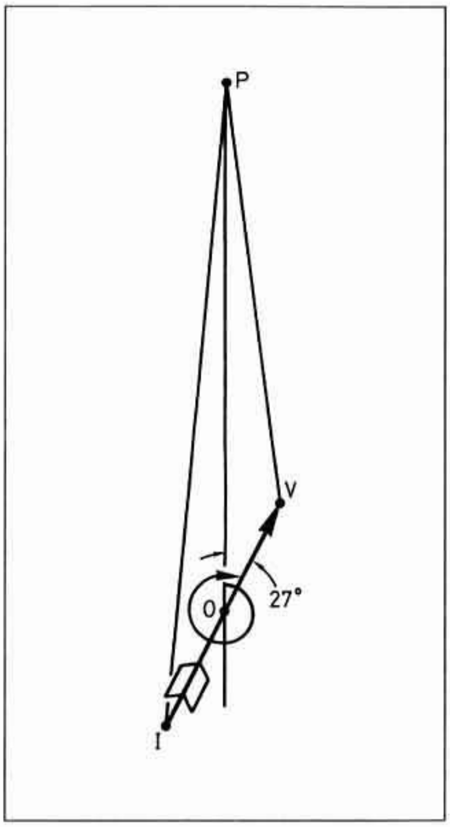

Fig 8: Vector diagram for first example.

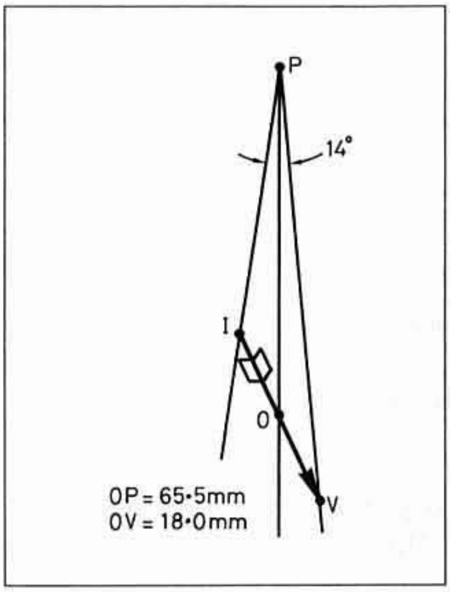

Fig 9: Alternative representation of line impedance.

tion allows the impedance at any other point to be found.

Two more formulas are required. They are standard AC formulas for series circuits with no special relevance to transmission lines. They are:

(i) $Z = \sqrt{(R^2 + X^2)}$
(ii) $\tan\theta = X/R$

(θ is actually the phase angle between the voltage and current.)

Suppose for example that the bridge measurements, made on 50Ω coaxial cable, were:

$R = 80\Omega$
$X = 20\Omega$ (inductive)

First find Z:

$$Z = \sqrt{(R^2 + X^2)} = \sqrt{(80^2 + 20^2)} = 82.5\Omega$$

Next find θ:

$$\tan\theta = X/R = 20/80 = 0.25$$

Enter 0.25 in the calculator and find the inverse tan.

$\theta = 14.0°$

PV and PI can now be drawn making the correct angle θ, and with their lengths adjusted so that $PV/PI = Z/Z_0$.

The simplest way of doing this is to make PV of length Z units, and PI of length Z_0 units. Proceed as follows (see **Fig 9**).

Mark a point 'P' near the top of the paper. Draw a line 'PV' inclined to the right of the vertical and of length 82.5 units.

Draw another line 'PI' of length 50 units to the left of PV and making an angle of 14° with PV. Note that if X had been capacitative, PV would have to be drawn on the left and PI on the right.

Join VI and mark the mid point 'O'. Put the arrowhead at V. Join PO. This gives the main axis. It will almost certainly be inclined to the vertical, but this is of no importance. Measure OP and OV.

$k = OV/OP$
$k = 0.27$

This gives an SWR of 1.7, which could have been checked if a suitable meter had been available.

It is not essential to make any measurements on this preliminary diagram. The aim of the exercise is to find the line impedance at some other point, probably at the junction with the antenna. All that matters is that the indicator is correctly drawn relative to the main axis. Providing PV and PI have been lightly drawn they can be erased, and the new position of the indicator put in after making the appropriate rotation. The second diagram is then constructed keeping OP and the indicator length unchanged.

ALTERING THE FEEDER LENGTH

MEASURING THE ANGLE between the indicator and the main axis on the preliminary diagram does give useful information about the position of the maxima and minima.

The indicator makes an angle of 155° with the main axis if measured from 12 o'clock, or 25° if measured from 6 o'clock. Thus moving down the line a distance equivalent to an anticlockwise rotation of 155° brings us to a voltage minimum. If the line could be cut at this point it would present a purely resistive impedance $Z_0/SWR\Omega$.

The actual length to be removed is L metres where

$L = 155\,v/2.4f$
(v is the velocity factor)

Similarly, extending the feeder by a distance equivalent to a 25° rotation would present a voltage maximum and a purely resistive load to the transmitter. The length of the extension (using identical feeder) would be 25v/2.4f metres.

THE MEANING OF THE DIAGRAM AND ITS LIMITATIONS

THE DIAGRAM IS IN FACT a vector (phasor) diagram, which correctly shows the current and voltage, and the phase difference between them at any point on the line.

PV is the voltage vector. PI is the current vector - or to be precise, the current vector scaled up by a factor Z_0.

θ is the phase angle between voltage and current.

Looking at the diagram, and picturing what happens as the indicator rotates, it becomes clear why the line current decreases as the voltage increases and vice versa, and why current minima and maxima occur with voltage maxima and minima respectively.

It is also becomes clear why the ratio of maximum to minimum voltage (or current), which is defined as the SWR, is also equal to $(1 + k)/(1 - k)$, as stated earlier.

There is one piece of information about voltage and current which the diagram does not supply. Without further modification it does not allow you to compare the phase of the voltage or current at one point on the line with the phase at another point. For instance, the phase of the voltage at any point on the line is in antiphase with the voltage at a point half a wavelength from it, and the same is true of currents. The diagram does not show this. According to the diagram everything repeats every half wavelength, whereas in fact the signs of the voltage and current reverse.

The reason for this is because of a simplification introduced into the drawing of the diagram. To draw the diagram in such a way as to preserve all the phase relationships, the indicator should only be rotated through an angle of $360(L/\lambda)$ to represent a distance L, whilst at the same time OP should be rotated through an equal angle in the opposite direction. This would preserve the correct phase relationship between voltages and currents at all points. However, it would be a needless and extremely inconvenient complication. The resultant angle between OP and the indicator would be exactly the same as in the standard method, and the diagram would be identical apart from being rotated on the paper. Reversing the sign of both voltage and current would have no effect on the impedance.

Two other simplifications have been made. First, it has been assumed that all lines are loss free. A lossy line will apparently have a lower value of k than the value at the termination. This effect increases with length of line.

Secondly, no account is taken of end corrections. These might cause slight errors in the case of wide-spaced lines.

SUMMARY OF SOME IMPORTANT FORMULAE

(1) $k = (SWR - 1)/(SWR + 1)$

(2) $SWR = (1 + k)/(1 - k) = V_{max}/V_{min} = I_{max}/I_{min}$

(3) Rotation of indicator for a distance $L = 720 (L/\lambda)$ degrees, or more conveniently:

(4) Rotation (degrees) = 2.4 Lf/Velocity factor

L in metres, f in Megahertz

(5) From the diagram $Z = Z_0(PV/PI)$
$R = Z\cos\theta$
$X = Z\sin\theta$

(6) $X/R = \tan\theta$

(7) $Z = \sqrt{(R^2 + X^2)}$

(8) In (5), (6) and (7), R and X are *series* components of Z. If the equivalent *parallel* components of the impedance Z are required, PV, PI and θ are measured and Z calculated as before.

The parallel resistive component = $Z/\cos\theta$
The parallel reactive component = $Z/\sin\theta$

The sign of the reactive component is the same for both the series and parallel cases.

APPENDIX

Deriving The Diagram

The potential at a point on a line may be represented as the resultant of two component potentials, one of V volts RMS due to a wave travelling from left to right (say), and one of kV volts due to a reflected wave travelling in the opposite direction.

Let there be a voltage maximum at some point 'A'. These two components are in phase at this point, giving a resultant of $V(1 + k)$ (**Fig A1** (top))

At a point distance 'L' to the right of A, the component due to the forward wave will lag on the voltage at A by an angle $a = 360L/\lambda$ degrees, whilst the component due to the reflected wave will lead the voltage at A by the same angle.

Fig A1 (bottom) shows the component and resultant voltages at the new point. The resultant magnitude of the voltage at any point on the line can be found in this way.

In what follows it becomes more convenient to keep the vector representing the forward voltage fixed and to rotate the other vector through the double angle 2a.

It is next necessary to find the magnitude of the current at any point, and also the phase angle between the voltage and current. The

current at any point may also be represented as the resultant of two components: the current due to the forward wave is of amplitude V/Z_0 and that due to the reflected wave is kV/Z_0. The forward wave current is in phase with the forward wave voltage.

The reflected wave current must be taken as being in antiphase with the reflected wave voltage.

This may be explained by considering a charged region moving to the right, and then a similarly charged region moving to the left. The potentials will have the same sign in both cases, but the currents must be given opposite signs due to the opposite directions of motion. It is now possible to draw a voltage parallelogram and a current parallelogram side by side (**Fig A2** (top)).

If the current parallelogram is scaled up by a factor Z_0, it becomes identical to the voltage parallelogram, except that the other diagonal is used (**Fig A2** (bottom)). The angle between the diagonals is the phase angle between the resultant current and voltage.

The diagram described in the main part of the article is derived directly from Fig A4, which is then inverted to give a closer similarity to a Smith chart.

SOLVING PROBLEMS WITHOUT SCALE DIAGRAMS

IF YOU ARE HANDY WITH a pocket calculator, it may be easier to calculate numerical results rather than draw a diagram accurately to scale. Even so, it is really essential to sketch rough diagrams to help you to monitor what you are doing.

Three new formulas are required in addition to those already introduced. These new formulas are slightly simplified by quoting them in 'normalised' or 'relative' impedance units. This means that impedances are given as multiples of Z_0.

Relative impedance =
(impedance in Ω)/Z_0 (in Ω).

Thus a resistance of 100Ω has a relative resistance of 2 'zednoughts' when using 50Ω feeder.

These 'normalised' quantities are here denoted by the appropriate small letter, whilst capital letters denote the same quantities measured in ohms.

$$z = Z/Z_0;\ r = R/Z_0;\ x = X/Z_0$$

These 'relative' or 'normalised' units are also employed by the Smith chart.

The three formulas are:

$$\text{(i)}\quad k = \sqrt{\frac{(1 - r)^2 + x^2}{(1 + r)^2 + x^2}}$$

$$\text{(ii)}\quad \tan\theta = x/r = \left(\frac{2k}{1 - k^2}\right)\sin\phi$$

$$\text{(iii)}\quad z = \sqrt{\frac{1 + k^2 - 2k\cos\phi}{1 + k^2 + 2k\cos\phi}}$$

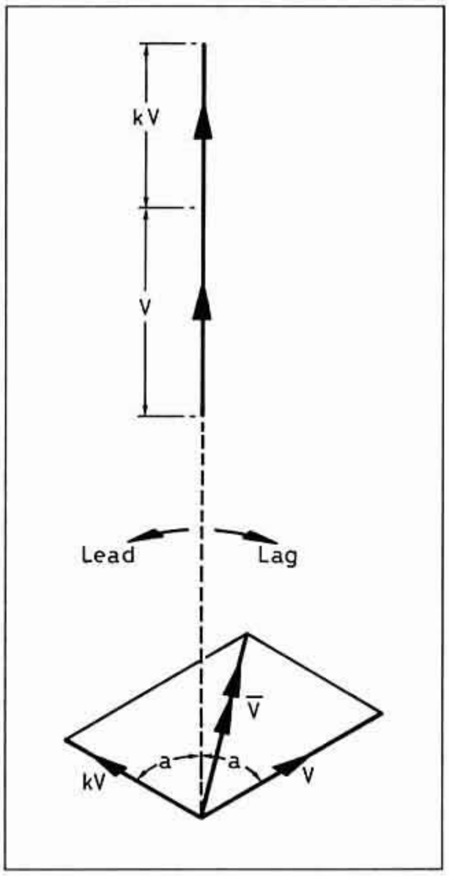

Fig A1: Forward and reflected voltages.

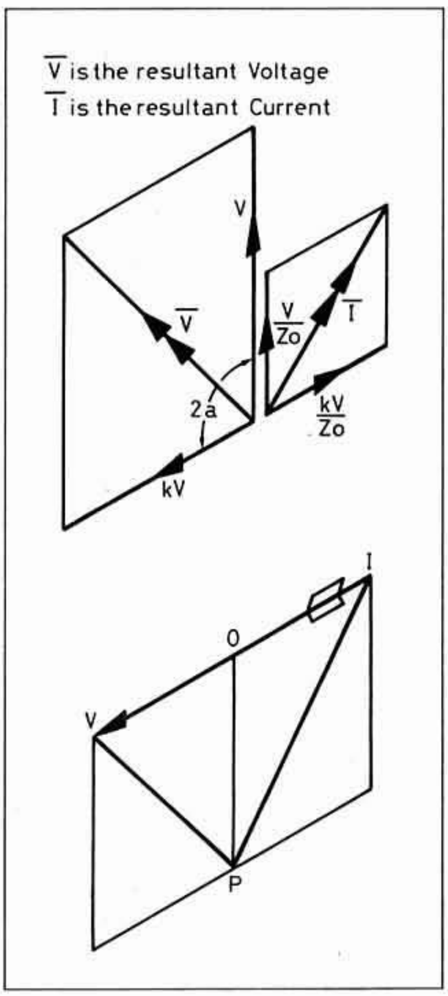

Fig A2: Voltage and current components.

Rearranging gives $\cos\phi = \left(\frac{1 + k^2}{2k}\right)\left(\frac{1 - z^2}{1 + z^2}\right)$

The angle ϕ is the angle between the indicator and the main axis and should be measured from 12 o'clock, not from 6 o'clock.

As an example, equations (i) and (ii) can be applied to solve the problem in the section 'Drawing the Diagram in Reverse'.

Use (i) to find k.

$$r = 80/50 = 1.6$$
$$x = 20/50 = 0.4$$

$$k = \sqrt{\frac{(1 - 1.6)^2 + 0.4^2}{(1 + 1.6)^2 + 0.4^2}}$$

$$k = 0.274$$

Use (ii) or (iii) to find ϕ.

$$\sin\phi = x/r\left(\frac{1 - k^2}{2k}\right)$$

$$= 0.422$$
$$\therefore \phi = 25° \text{ (using 'inverse sine')}$$

There are in fact two significant values of ϕ corresponding to a given value of $\sin\phi$, one less than 90° and the other greater. To obtain the latter value, the smaller angle must be subtracted from 180°. The calculator only gives the smaller value, 25° in this case, so the other possible value is 155°.

This ambiguity does not arise when looking up inverse cosines. Luckily it is easy to find the correct answer. Simply evaluate z using:

$$z = \sqrt{(r^2 + x^2)}$$

If z is greater than unity the greater angle is the correct one; if z is smaller than unity choose the smaller angle. If you sketch one or two diagrams, remembering that ϕ is defined as measured from 12 o'clock, the reason should become clear. Applying the rule to the present case we see that Z is greater than Z_0, z is greater than unity, so ϕ = 155° is the correct answer.

The impedance z at any other point at a distance L metres from the first may be found.

The angular rotation must be calculated using

$$\text{angle in degrees} = 2.4\ Lf/v$$

The new value of ϕ must then be found by rotating the indicator from its present position through the above angle in the appropriate direction. A rough sketch is a help here.

The new value of z is found using the equation (iii).

The resistive and reactive components of z are obtained by using equation (ii) to find the new value of θ and then using

$$r = z\cos\theta$$
$$x = z\sin\theta$$

Again you will need your rough sketch to see whether x is inductive or capacitative.